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Impurity operators in RSOS models

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Abstract. We give a construction of impurity operators in the 'algebraic analysis' picture of RSOS models. Physically, these operators are half-infinite insertions of certain fusion-RSOS Boltzmann weights. They are the face analogue of insertions of higher-spin lines in vertex models. Mathematically, they are given in terms of intertwiners of $U_q(\widehat{\mathfrak{sl}}_2)$ modules. We present a detailed perturbation theory check of the conjectural correspondence between the physical and mathematical constructions for a particular simple example.

1. Introduction

The 'algebraic analysis' approach to solvable lattice models was developed by the Kyoto group in the 1990s [1]. The key feature of this approach is to identify the half-infinite space on which the corner transfer matrix acts with an infinite-dimensional module of the underlying non-Abelian symmetry algebra of the lattice model. The simplest example is the anti-ferromagnetic six-vertex model, in which the half-infinite space is identified with $V(\Lambda_i)$, a level-one highest-weight module of the algebra $U_q(\widehat{\mathfrak{sl}}_2)$ [1, 2]. The choice of the subscript $i \in \{0, 1\}$ corresponds to the choice of one of the two possible anti-ferromagnetic boundary conditions. A somewhat more complicated example is that of RSOS models [3]. Here, the half-infinite space is identified with the space $\Omega_{\xi,\eta;\lambda}$ that occurs in the decomposition of the tensor product of $U_q(\widehat{\mathfrak{sl}}_2)$ highest-weight modules

$$V(\xi) \otimes V(\eta) \simeq \bigoplus_{\lambda} V(\lambda) \otimes \Omega_{\xi,\eta;\lambda}$$

where ξ and η are level-$(k-n)$ and level-n dominant integral weights, and the sum is over all level-k dominant integral weights (see [3] and below for more details). Again, the choice of ξ, η and λ in $\Omega_{\xi,\eta;\lambda}$ corresponds to the choice of boundary conditions for the lattice model.

The other main step in the algebraic analysis approach is to identify the half-infinite transfer matrices of the lattice models with certain intertwiners, or vertex operators, of the symmetry algebra. For the six-vertex model, the half-infinite transfer matrix is identified with the $U_q(\widehat{\mathfrak{sl}}_2)$ intertwiner

$$\Phi_{\Lambda_i}^{\Lambda_{1-i}V^{(1)}}(\zeta) : V(\Lambda_i) \to V(\Lambda_{1-i}) \otimes V_\zeta^{(1)}$$

where $V_\zeta^{(1)}$ is a spin-$\frac{1}{2}$ $U_q(\widehat{\mathfrak{sl}}_2)$ evaluation module. For the RSOS case, the situation is again slightly more complicated. If λ and λ' are level-k dominant integral weights, then the $U_q(\widehat{\mathfrak{sl}}_2)$ intertwiner

$$\Phi_{\lambda}^{\lambda' V^{(n)}}(\zeta) : V(\lambda) \to V(\lambda') \otimes V_\zeta^{(n)}$$

exists if and only if $1 \leqslant n \leqslant k$ and the pair (λ, λ') is 'n admissible' as defined by (2.1) below. Consider the intertwiner

$$V(\xi) \otimes V(\eta) \xrightarrow{1\otimes\Phi_\eta^{\sigma(\eta)V^{(n)}}(\zeta)} V(\xi) \otimes V(\sigma(\eta)) \otimes V_\zeta^{(n)}$$

where $\sigma(a\Lambda_1 + b\Lambda_0) = (b\Lambda_1 + a\Lambda_0)$. Under the isomorphism (1), we can identify

$$1 \otimes \Phi_\eta^{\sigma(\eta)V^{(n)}} = \bigoplus_{(\lambda,\lambda')} \Phi_\lambda^{\lambda' V^{(n)}}(\zeta) \otimes X_\lambda^{\lambda'}(\zeta)$$

where the sum is over all n-admissible pairs (λ, λ') of level-k dominant integral weights. This identification defines the operator

$$X_\lambda^{\lambda'}(\zeta) : \Omega_{\xi,\eta;\lambda} \to \Omega_{\xi,\sigma(\eta);\lambda'}$$

(see (2.11)). It is $X_\lambda^{\lambda'}(\zeta)$ that is identified with the half-infinite transfer matrix of the RSOS lattice model [3].

The role of impurity operators in the six-vertex model was considered in [4,5]. For vertex models, the term impurity operator refers to the half-infinite transfer matrix corresponding to the insertion of a spin-$\frac{n}{2}$ line into a spin-$\frac{1}{2}$ six-vertex model. In [5], this operator was identified with the $U_q(\widehat{\mathfrak{sl}}_2)$ intertwiner

$$\Phi_{\Lambda_i}^{(n-1,n)} : V_\zeta^{(n-1)} \otimes V(\Lambda_i) \to V(\Lambda_{1-i}) \otimes V_\zeta^{(n)}$$

(which exists for all $n > 1$).

In this paper, we shall consider analogous impurity operators in RSOS models. The physical impurity operator corresponds to the half-infinite insertion of $W_k^{(m,n)}$ RSOS weights (see section 3.1) into a lattice made up of $W_k^{(n,n)}$ weights. The mathematical object with which this lattice operator will be identified is defined in terms of the composition

$$\begin{aligned} V(\xi) \otimes V(\eta) &\xrightarrow{\Phi_\xi^{\xi' V^{(m-n)}}(\zeta)} V(\xi') \otimes V_\zeta^{(m-n)} \otimes V(\eta) \\ &\xrightarrow{\Phi_\eta^{(m-n,m)}(\zeta)} V(\xi') \otimes V(\sigma(\eta)) \otimes V_\zeta^{(m)} \end{aligned}$$

where $k \geqslant m > n \geqslant 1$ (see (2.4) for a definition of $\Phi_\eta^{(m-n,m)}(\zeta)$). Under the isomorphism (1), we identify

$$\Phi_\eta^{(m-n,m)}(\zeta) \circ \Phi_\xi^{\xi' V^{(m-n)}}(\zeta) = \bigoplus_{(\lambda,\lambda')} \Phi_\lambda^{\lambda' V^{(m)}}(\zeta) \otimes Z_{\xi\ \lambda;m}^{\xi'\lambda'}(\zeta)$$

where again the sum is over all all n-admissible pairs (λ, λ'). This equality defines the operator

$$Z_{\xi\ \lambda;m}^{\xi'\lambda'}(\zeta) : \Omega_{\xi,\eta;\lambda} \to \Omega_{\xi',\sigma(\eta);\lambda'}.$$

It is this that we shall identify with the RSOS impurity operator (a statement of the conjectural identification is given in (3.9)).

The plan of this paper is as follows: in section 2, we define the necessary $U_q(\widehat{\mathfrak{sl}}_2)$ intertwiners and give some of their properties. In section 3, we recall some of the details of the algebraic analysis picture of RSOS models and give our precise conjecture about the realization of impurity operators in this picture. We give the details of a perturbation theory check of this conjecture in section 4. We present a brief discussion of some possible future avenues of research opened by this work in section 5. In appendix A, we give the solution of the q-KZ equation for certain matrix elements of intertwiners and use this solution in order to compute their commutation relations. In appendix B, we give a proof of the commutation relations of another type of intertwiner. Finally, we list some formulae for the perturbative action of our different intertwiners in appendix C.

2. Properties of $U_q(\widehat{\mathfrak{sl}_2})$ intertwiners

2.1. Preliminaries

In this section, we shall define the $U_q(\widehat{\mathfrak{sl}_2})$ intertwiners we need in our discussion, and give some of their properties. Let us first recall a few details about $U_q(\widehat{\mathfrak{sl}_2})$. (See, e.g., [6] for a fuller account—the only significant difference from our notation is that we use a different evaluation module. Note, also, that although we use the notation $U_q(\widehat{\mathfrak{sl}_2})$, we are actually referring to the subalgebra generated by e_i, f_i, t_i $(i = 0, 1)$.) A weight lattice $P_+ = \mathbb{Z}_{\geqslant 0}\Lambda_0 \oplus \mathbb{Z}_{\geqslant 0}\Lambda_1$ occurs in the definition of $U_q(\widehat{\mathfrak{sl}_2})$. Let h_0 and h_1 denote the basis vectors for the lattice dual to P_+, with $\langle h_i, \Lambda_j\rangle = \delta_{i,j}$. Define the level $k \in \mathbb{Z}_{\geqslant 0}$ weight $\lambda_a^{(k)} \in P_+$ by

$$\lambda_a^{(k)} = a\Lambda_1 + (k-a)\Lambda_0 \qquad a \in \{0, 1, \ldots, k\}.$$

Let P_k^0 be the set of such weights, i.e.

$$P_k^0 = \{\lambda_a^{(k)} \mid a \in \{0, 1, \ldots, k\}\}$$

and define the function $\sigma : P_+ \to P_+$ by

$$\sigma(a\Lambda_1 + b\Lambda_0) = b\Lambda_1 + a\Lambda_0.$$

We shall also use the notation

$$p = q^{2(k+2)} \qquad s = \frac{1}{2(k+2)} \qquad \bar{\rho} = (\Lambda_1 - \Lambda_0).$$

Suppose we choose an integer N such that $k \geqslant N \geqslant 0$. Then a pair of weights $(\lambda_a^{(k)}, \lambda_b^{(k)})$ is said to be 'N admissible' if:

$$\begin{aligned} &\text{(i)} && a - b \in \{N, N-2, \ldots, -N\} \\ &\text{(ii)} && a + b \in \{2k - N, 2k - N - 1, \ldots, N\}. \end{aligned} \tag{2.1}$$

In the case $N = 0$, we have $a = b$. When $N = 1$, the second condition follows from the first. It is useful to introduce the notation $A_k^{(N)}$ for the set of admissible pairs, i.e.

$$A_k^{(N)} = \{(\lambda, \lambda') \in P_k^0 \times P_k^0 \mid (\lambda, \lambda') \text{ are } N\text{-admissible}\}.$$

Note that if $(\lambda_1, \lambda_1') \in A_{k_1}^{(N_1)}$ and $(\lambda_2, \lambda_2') \in A_{k_2}^{(N_2)}$, then it follows that $(\lambda_1 + \lambda_2, \lambda_1' + \lambda_2') \in A_{k_1+k_2}^{(N_1+N_2)}$.

We shall use two types of $U_q(\widehat{\mathfrak{sl}_2})$ module: irreducible highest-weight modules $V(\lambda)$ and evaluation modules $V_\zeta^{(N)}$. The irreducible highest-weight module $V(\lambda)$ is generated by a highest-weight vector v_λ, defined by $e_i v_\lambda = 0$, $f_i^{\langle h_i,\lambda\rangle+1} v_\lambda = 0$, for $i \in \{0, 1\}$. We use the principally specialized spin-$\frac{N}{2}$ evaluation module $V_\zeta^{(N)}$ defined, in terms of weight vectors $u_i^{(N)}$ $(i = 0, 1, \ldots, N)$, in section 3.1 of [5].

We will also need the R-matrix, namely the $U_q(\widehat{\mathfrak{sl}_2})$ intertwiner

$$R^{(M,N)}(\zeta_1/\zeta_2) : V_{\zeta_1}^{(M)} \otimes V_{\zeta_2}^{(N)} \to V_{\zeta_2}^{(N)} \otimes V_{\zeta_1}^{(M)}.$$

The normalization is fixed by $R^{(M,N)}(\zeta) = \bar{R}^{(M,N)}(\zeta)/\kappa^{(M,N)}(\zeta)$, where

$$\begin{aligned} &\bar{R}^{(M,N)}(\zeta)(u_0^{(M)} \otimes u_0^{(N)}) = (u_0^{(N)} \otimes u_0^{(M)}) \qquad \text{and} \\ &\kappa^{(M,N)}(\zeta) = \zeta^{\min(M,N)} \frac{(q^{2+M+N}\zeta^2;\, q^4)_\infty (q^{2+|M-N|}\zeta^{-2};\, q^4)_\infty}{(q^{2+M+N}\zeta^{-2};\, q^4)_\infty (q^{2+|M-N|}\zeta^2;\, q^4)_\infty}. \end{aligned} \tag{2.2}$$

(We use the standard notation $(a; b)_\infty = \prod_{n=0}^{\infty}(1 - a\, b^n)$.) This is the normalization that ensures crossing and unitarity for the R-matrix—see [7] † (this normalization is also the one that would give the vertex model with $R^{(M,N)}(\zeta)$ Boltzmann weights a partition function per site equal to one).

† Published in a special edition of *AJM* dedicated to Professor M Sato on his 70th birthday.

2.2. Intertwiners

We shall make use of the following two types of $U_q(\widehat{\mathfrak{sl}}_2)$ intertwiner:

$$\Phi_{\lambda}^{\lambda' V^{(N)}}(\zeta) : V(\lambda) \to V(\lambda') \otimes V_{\zeta}^{(N)} \qquad (\lambda, \lambda') \in A_k^{(N)} \quad N \in \{1, 2, \ldots, k\} \tag{2.3}$$

$$\Phi_{\lambda}^{(N,N+k)}(\zeta) : V_{\zeta}^{(N)} \otimes V(\lambda) \to V(\sigma(\lambda)) \otimes V_{\zeta}^{(N+k)} \qquad \lambda \in P_k^0 \quad N \in \mathbb{Z}_{>0}. \tag{2.4}$$

It is shown in [8] that $\Phi_{\lambda}^{\lambda' V^{(N)}}(\zeta)$ exists and is unique up to a normalization if and only if (λ, λ') is an N-admissible pair. The existence and uniqueness of $\Phi_{\lambda}^{(N,N+k)}(\zeta)$ is shown in [7] (the $k = 1$ operator was first introduced by Nakayashiki in [4]). We fix the normalization of $\Phi_{\lambda}^{\lambda' V^{(N)}}(\zeta)$ by the requirement

$$\Phi_{\lambda}^{\lambda' V^{(N)}}(\zeta) : v_{\lambda} \longmapsto v_{\lambda'} \otimes u_j^{(N)} + \cdots \quad \text{where} \quad \lambda = \lambda' + (N - 2j)\bar{\rho}. \tag{2.5}$$

Here, $\cdots$ means terms involving $F v_{\lambda'}$, where F is some product of f_0 and f_1 generators. The normalization of $\Phi_{\lambda}^{(N,N+k)}(\zeta)$ is that given in section 5 of [7].

Now, we shall give the commutation relations of the two types of intertwiner (2.3) and (2.4). In [8], Frenkel and Reshetikhin showed that the commutation relations of (2.3) take the form

$$R^{(M,N)}(\zeta)\Phi_{\mu}^{\nu V^{(M)}}(\zeta_1)\Phi_{\lambda}^{\mu V^{(N)}}(\zeta_2) = \sum_{\mu'} \Phi_{\mu'}^{\nu V^{(N)}}(\zeta_2)\Phi_{\lambda}^{\mu' V^{(M)}}(\zeta_1) C_k^{(N,M)} \left(\begin{array}{cc} \lambda & \mu \\ \mu' & \nu \end{array} \middle| \zeta \right) \tag{2.6}$$

where $\zeta = \zeta_1/\zeta_2$, the sum is over $\{\mu' \in P_k^0 \,|\, (\nu, \mu') \in A_k^{(N)}, (\mu', \lambda) \in A_k^{(M)}\}$ and the connection coefficients $C_k^{(N,M)}$ satisfy the Yang–Baxter equation in its face formulation. As a special case, we have

$$C_k^{(k,k)} \left(\begin{array}{cc} \lambda & \mu \\ \mu' & \nu \end{array} \middle| \zeta \right) = \delta_{\lambda,\nu}\delta_{\mu,\mu'}\delta_{\mu,\sigma(\nu)}$$

(see [6]). In appendix A, we solve the q-KZ equation to obtain the explicit form (A.7)–(A.12) of the coefficients $C_k^{(N,1)}$ and $C_k^{(1,N)}$. In appendix B, we prove that the commutation relations of (2.4) are given by

$$R^{(N+k,N+k)}(\zeta)\Phi_{\sigma(\lambda)}^{(N,N+k)}(\zeta_1)\Phi_{\lambda}^{(N,N+k)}(\zeta_2) = \Phi_{\sigma(\lambda)}^{(N,N+k)}(\zeta_2)\Phi_{\lambda}^{(N,N+k)}(\zeta_1)R^{(N,N)}(\zeta). \tag{2.7}$$

2.3. Operators on the space $\Omega_{\xi,\eta;\lambda}$

Fix $\xi \in P_{k-n}^0$ and $\eta \in P_n^0$ with $k > n \geqslant 1$. Following [3] and [9], we consider the decomposition

$$V(\xi) \otimes V(\eta) \simeq \bigoplus_{\lambda \in P_k^0} V(\lambda) \otimes \Omega_{\xi,\eta;\lambda}. \tag{2.8}$$

Here $\Omega_{\xi,\eta;\lambda}$ denotes the space of highest vectors

$$\Omega_{\xi,\eta;\lambda} = \{v \in V(\xi) \otimes V(\eta) \,|\, e_i v = 0,\ t_i v = q^{\langle h_i, \lambda \rangle} v\}.$$

The existence of this decomposition allows us to use the intertwiners (2.3) and (2.4) in order to define certain operators on $\Omega_{\xi,\eta;\lambda}$. Namely, we define

$$X_{\lambda}^{\lambda'}(\zeta) : \Omega_{\xi,\eta;\lambda}(\zeta) \to \Omega_{\xi,\sigma(\eta);\lambda'} \qquad \text{for} \quad (\lambda, \lambda') \in A_k^{(n)} \tag{2.9}$$

$$Z_{\xi\,\lambda;m}^{\xi'\lambda'}(\zeta) : \Omega_{\xi,\eta;\lambda} \to \Omega_{\xi',\sigma(\eta);\lambda'} \qquad \text{for} \quad (\lambda, \lambda') \in A_k^{(m)} \quad (\xi, \xi') \in A_{k-n}^{(m-n)} \quad k \geqslant m > n \tag{2.10}$$

via the identifications

$$\Phi_\eta^{\sigma(\eta)V^{(n)}}(\zeta) = \bigoplus_{(\lambda,\lambda')\in A_k^{(n)}} \Phi_\lambda^{\lambda' V^{(n)}}(\zeta) \otimes X_\lambda^{\lambda'}(\zeta) \tag{2.11}$$

$$\Phi_\eta^{(m-n,m)}(\zeta) \circ \Phi_\xi^{\xi' V^{(m-n)}}(\zeta) = \bigoplus_{(\lambda,\lambda')\in A_k^{(m)}} \Phi_\lambda^{\lambda' V^{(m)}}(\zeta) \otimes Z_{\xi\ \lambda;m}^{\xi'\lambda'}(\zeta). \tag{2.12}$$

It should be clear from the subscripts on which part of $V(\xi) \otimes V(\eta)$ the operators on the left-hand side act. In section 3.3, we shall use a single notation for both (2.9) and (2.10), by defining $Z_{\xi\ \lambda;n}^{\xi\lambda'}(\zeta)$ by $Z_{\xi\ \lambda;n}^{\xi\lambda'}(\zeta) = X_\lambda^{\lambda'}(\zeta)$.

The commutation relations of X and Z follow from their definitions (2.11) and (2.12), and from (2.6) and (2.7). We find that, acting on $\Omega_{\xi,\eta;\lambda}$, we have

$$\sum_{\tilde\lambda\in P_k^0} C_k^{(n,n)}\left(\begin{matrix}\lambda & \tilde\lambda \\ \lambda' & \lambda''\end{matrix}\middle|\zeta_1/\zeta_2\right) X_{\tilde\lambda}^{\lambda''}(\zeta_1) X_\lambda^{\tilde\lambda}(\zeta_2) = X_{\lambda'}^{\lambda''}(\zeta_2) X_\lambda^{\lambda'}(\zeta_1)$$

$$\sum_{\tilde\lambda\in P_k^0} C_k^{(m,m)}\left(\begin{matrix}\lambda & \tilde\lambda \\ \lambda' & \lambda''\end{matrix}\middle|\zeta_1/\zeta_2\right) Z_{\xi'\ \tilde\lambda;m}^{\xi''\lambda''}(\zeta_1) Z_{\xi\ \lambda;m}^{\xi\tilde\lambda}(\zeta_2)$$

$$= \sum_{\tilde\xi\in P_{k-n}^0} Z_{\tilde\xi\ \lambda';m}^{\xi''\lambda''}(\zeta_2) Z_{\xi\ \lambda;m}^{\tilde\xi\lambda'}(\zeta_1) C_{k-n}^{(m-n,m-n)}\left(\begin{matrix}\xi & \xi' \\ \tilde\xi & \xi''\end{matrix}\middle|\zeta_1/\zeta_2\right).$$

3. The algebraic analysis picture of RSOS models

3.1. The RSOS lattice model

Let us define lattice Boltzmann weights $W_k^{(m,n)}$ with $k \geqslant m, n \geqslant 1$ by

$$W_k^{(m,n)}\left(\begin{matrix}\lambda & \mu \\ \mu' & \nu\end{matrix}\middle|\zeta\right) = C_k^{(n,m)}\left(\begin{matrix}\nu & \mu \\ \mu' & \lambda\end{matrix}\middle|\zeta\right)$$

where $(\lambda,\mu), (\mu',\nu) \in A_k^{(m)}$ and $(\lambda,\mu'), (\mu,\nu) \in A_k^{(n)}$, and where the connection coefficients $C_k^{(n,m)}$ are defined via (2.6). Then, it follows from (2.6) and from the Yang–Baxter equation and unitarity property of $R^{(m,n)}(\zeta)$ (see [7]) that $W_k^{(m,n)}$ has the analogous face properties

$$\sum_{\nu\in P_k^0} W_k^{(n,\ell)}\left(\begin{matrix}\alpha & \nu \\ \mu & \lambda\end{matrix}\middle|\zeta_2/\zeta_3\right) W_k^{(m,n)}\left(\begin{matrix}\alpha & \beta \\ \nu & \gamma\end{matrix}\middle|\zeta_1/\zeta_2\right) W_k^{(m,\ell)}\left(\begin{matrix}\nu & \gamma \\ \lambda & \delta\end{matrix}\middle|\zeta_1/\zeta_3\right)$$

$$= \sum_{\nu\in P_k^0} W_k^{(m,\ell)}\left(\begin{matrix}\alpha & \beta \\ \mu & \nu\end{matrix}\middle|\zeta_1/\zeta_3\right) W_k^{(m,n)}\left(\begin{matrix}\mu & \nu \\ \lambda & \delta\end{matrix}\middle|\zeta_1/\zeta_2\right)$$

$$\times W_k^{(n,\ell)}\left(\begin{matrix}\beta & \gamma \\ \nu & \delta\end{matrix}\middle|\zeta_2/\zeta_3\right)$$

$$\sum_{\mu'\in P_k^0} W_k^{(m,n)}\left(\begin{matrix}\lambda & \mu \\ \mu' & \nu\end{matrix}\middle|\zeta\right) W_k^{(n,m)}\left(\begin{matrix}\lambda & \mu' \\ \alpha & \nu\end{matrix}\middle|\zeta^{-1}\right) = \delta_{\mu,\alpha}.$$

We can prove some additional properties of $W_k^{(n,1)}$ and $W_k^{(1,n)}$ by making use of the explicit formulae for these weights given in appendix A. The first property relates $W_k^{(n,1)}$ and $W_k^{(1,n)}$:

$$W_k^{(n,1)}\left(\begin{matrix}\lambda & \mu \\ \mu' & \nu\end{matrix}\middle|\zeta\right) = W_k^{(1,n)}\left(\begin{matrix}\nu & \mu \\ \mu' & \lambda\end{matrix}\middle|\zeta\right). \tag{3.1}$$

The second property is that of crossing symmetry:

$$W_k^{(n,1)}\left(\begin{array}{cc}\lambda & \mu \\ \mu' & \nu\end{array}\middle| -q^{-1}\zeta\right) = \frac{G(\lambda,\mu')}{G(\mu,\nu)} W_k^{(1,n)}\left(\begin{array}{cc}\mu' & \lambda \\ \nu & \mu\end{array}\middle| \zeta^{-1}\right) \tag{3.2}$$

where

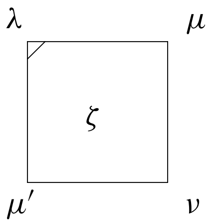
$$G(\lambda_a^{(k)},\lambda_{a+1}^{(k)}) = \frac{\Gamma_p(1-2s(a+1))}{\Gamma_p(1-2s(a+2))}$$

$$G(\lambda_a^{(k)},\lambda_{a-1}^{(k)}) = \frac{\Gamma_p(2s(a+1))}{\Gamma_p(2sa)}.$$

Here, Γ_p is the 'q–gamma' function defined in (A.10), and p and s are as defined in section 2.1. We anticipate that formulae similar to (3.1) and (3.2) will hold for the general $W_k^{(m,n)}$.

We shall define our lattice model by associating a Boltzmann weight $W_k^{(m,n)}$ with a configuration of P_k^0 weights around a face in the following way:

$$W_k^{(m,n)}\left(\begin{array}{cc}\lambda & \mu \\ \mu' & \nu\end{array}\middle| \zeta\right) \sim$$



Here, one corner is marked in order to give an orientation to the diagram.

The partition function of our lattice model will be a weighted sum over the configurations of the weights at the corners of faces. In order to specify this partition function in the infinite-volume limit, we must specify the boundary conditions for these configurations at large distances from the centre of the lattice. We will choose these boundary configurations such that the associated Boltzmann weights are maximal. Let us now fix q and ζ such that $0 < -q < \zeta^{-1} < 1$. Then from the explicit formula (A.7) we find that the largest Boltzmann weights $W_k^{(n,1)}$, $n \geqslant 1$, are those of the form

$$W_k^{(n,1)}\left(\begin{array}{cc}\xi'+\Lambda_i & \xi+\Lambda_{1-i} \\ \xi'+\Lambda_{1-i} & \xi+\Lambda_i\end{array}\middle| \zeta\right) \qquad \text{with} \quad (\xi,\xi') \in A_{k-1}^{(n-1)}. \tag{3.3}$$

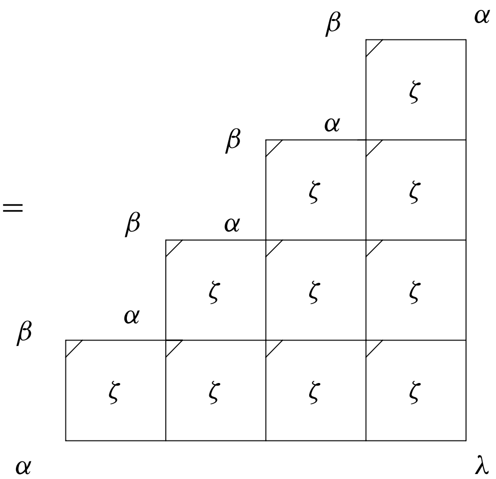
We assume, by extension, that when $k \geqslant m \geqslant n \geqslant 1$, the largest weights are those of the form

$$W_k^{(m,n)}\left(\begin{array}{cc}\xi'+\eta & \xi+\sigma(\eta) \\ \xi'+\sigma(\eta) & \xi+\eta\end{array}\middle| \zeta\right) \qquad \text{with} \quad (\xi,\xi') \in A_{k-n}^{(m-n)} \quad \eta \in P_n^0. \tag{3.4}$$

Now, following the approach to RSOS models described in [3] and [10], we consider the (n,n) RSOS lattice model, that is, the RSOS model constructed in terms of $W_k^{(n,n)}$, $n \geqslant 1$, weights. The boundary conditions will be labelled by a pair of weights $(\xi,\eta) \in P_{k-n}^0 \times P_n^0$ in the following way: if the position of the central weight is labelled 1, then we consider weight configurations such that, beyond a large but finite number of sites out from the centre, the weights at odd positions (along the vertical or horizontal directions) are fixed to be $\xi+\eta$, and the weights at even positions are fixed to be $\xi+\sigma(\eta)$.

The north-west corner transfer matrix $A_{\rm NW}^{(\xi,\eta;\lambda)}(\zeta)$ with this boundary condition, and with the centre weight fixed to $\lambda \in P_k^0$, is represented graphically by

$$A_{\rm NW}^{(\xi,\eta;\lambda)}(\zeta) =$$



$$\alpha = \xi + \eta \qquad \beta = \xi + \sigma(\eta).$$

Let $\mathcal{H}_{\xi,\eta;\lambda}$ denote the space of eigenstates of $A_{\rm NW}^{(\xi,\eta;\lambda)}(\zeta)$ in the infinite-volume limit, such that $A_{\rm NW}^{(\xi,\eta;\lambda)}(\zeta) : \mathcal{H}_{\xi,\eta;\lambda} \to \mathcal{H}_{\xi,\eta;\lambda}$. Let $|p\rangle$ denote a restricted path

$$|p\rangle = (\ldots, p(3), p(2), p(1)) \qquad \text{with} \quad (p(\ell+1), p(\ell)) \in A_k^{(n)} \qquad \text{for} \quad \ell \geqslant 1.$$

Then, $\mathcal{H}_{\xi,\eta;\lambda}$ will be formally spanned by the path space $\mathcal{P}_{\xi,\eta;\lambda}$ defined by

$$\mathcal{P}_{\xi,\eta;\lambda} = \{|p\rangle \mid p(\ell) = \xi + \sigma^{l-1}(\eta), \ell \geqslant r > 1,\ p(1) = \lambda\}.$$

3.2. The identification of $\Omega_{\xi,\eta;\lambda}$ and $\mathcal{H}_{\xi,\eta;\lambda}$

Let us first introduce some extra notation. Define $|p_{\xi,\eta}\rangle$ to be the 'ground-state' path in $P_{\xi,\eta;\,\xi+\eta}$ given by

$$|p_{\xi,\eta}\rangle = (\ldots, p_{\xi,\eta}(3), p_{\xi,\eta}(2), p_{\xi,\eta}(1)) \qquad \text{where} \quad p_{\xi,\eta}(\ell) = \xi + \sigma^{\ell-1}(\eta).$$

Also, define $v_{\xi,\eta} = v_\xi \otimes v_\eta \in \Omega_{\xi,\eta;\,\xi+\eta}$.

A map $\iota : \Omega_{\xi,\eta;\,\lambda} \to \mathcal{H}_{\xi,\eta;\,\lambda}$ is given in [3]. In our notation, this map is given by

$$\iota(v) = \sum_{|p\rangle \in \mathcal{P}_{\xi,\eta;\,\lambda}} c(p, v)|p\rangle \tag{3.5}$$

where

$$c(p, v) = \lim_{\ell\to\infty} \frac{c^\ell(p, v)}{c^\ell(p_{\xi,\eta}, v_{\xi,\eta})} \tag{3.6}$$

$$c^\ell(p, v) = \langle v_{\xi,\sigma^\ell(\eta)}| X_{p(\ell)}^{p_{\xi,\eta}(\ell+1)}(1) \ldots X_{p(2)}^{p(3)}(1) X_\lambda^{p(2)}(1)|v\rangle. \tag{3.7}$$

It is a conjecture that (3.6) converges.

3.3. The half-transfer matrix and impurity operators

First, we define the finite path space ${}_N\mathcal{P}_{\xi,\eta;\lambda}$ by

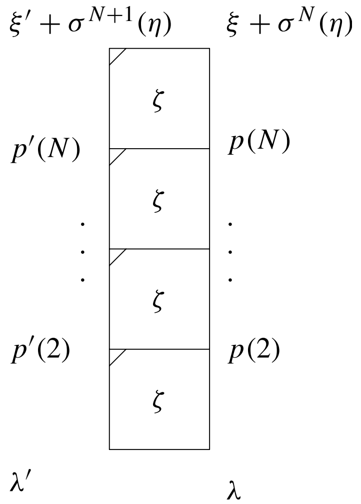
$$\begin{aligned}{}_N\mathcal{P}_{\xi,\eta;\lambda} = \{&(p(N+1), p(N), \ldots, p(1)) \mid (p(\ell+1), p(\ell)) \in A_k^{(n)},\\ &p(N+1) = \xi + \sigma^N(\eta),\ p(1) = \lambda\}.\end{aligned}$$

Let ${}_N\mathcal{H}_{\xi,\eta;\lambda}$ denote the vector space spanned by ${}_N\mathcal{P}_{\xi,\eta;\lambda}$, and define ρ_N to be the projection operator $\rho_N : \mathcal{H}_{\xi,\eta;\lambda} \to {}_N\mathcal{H}_{\xi,\eta;\lambda}$. Now we define the operator ${}_N Z^{\xi'\lambda'}_{\xi\ \lambda;m}(\zeta)$ by

$$
\begin{aligned}
&{}_N Z^{\xi'\lambda'}_{\xi\ \lambda;m}(\zeta) : {}_N\mathcal{H}_{\xi,\eta;\lambda} \to {}_N\mathcal{H}_{\xi',\sigma(\eta);\lambda'} \\
&\qquad\qquad \text{for} \quad (\lambda,\lambda') \in A^{(m)}_k \quad (\xi,\xi') \in A^{(m-n)}_{k-n} \quad k \geqslant m \geqslant n \geqslant 1 \\
&{}_N Z^{\xi'\lambda'}_{\xi\ \lambda;m}(\zeta)|p\rangle = \sum_{|p'\rangle \in {}_N\mathcal{P}_{\xi',\sigma(\eta);\lambda'}} \prod_{\ell=1}^{N} W^{(m,n)}\left(\begin{matrix} p'(\ell+1) & p(\ell+1) \\ p'(\ell) & p(\ell) \end{matrix} \middle| \zeta \right) |p'\rangle.
\end{aligned}
$$

Graphically, this operator is represented by

$${}_N Z^{\xi'\lambda'}_{\xi\ \lambda;m}(\zeta) = \tag{3.8}$$



Let $|v\rangle \in \Omega_{\xi,\eta;\lambda}$. Then our conjecture for the realization of ${}_N Z^{\xi'\lambda'}_{\xi\ \lambda;m}(\zeta)$ in the algebraic analysis picture of RSOS models is

$$\lim_{N\to\infty} \frac{1}{f^{(m,n)}_N(\zeta,q)}\, {}_N Z^{\xi'\lambda'}_{\xi\ \lambda;m}(\zeta) \circ \rho_N \circ \iota|v\rangle = \iota \circ Z^{\xi'\lambda'}_{\xi\ \lambda;m}(\zeta)|v\rangle \tag{3.9}$$

where the function $f^{(m,n)}_N(\zeta,q)$ is a series in q, whose coefficients are Laurent polynomials in ζ (this function may also depend upon the values of ξ, η, λ, ξ' and λ'). $Z^{\xi'\lambda'}_{\xi\ \lambda;m}(\zeta)$ is defined by (2.11) and (2.12) (with $Z^{\xi\lambda'}_{\xi\ \lambda;n}(\zeta) \equiv X^{\lambda'}_{\lambda}(\zeta)$).

When $m = n$, this conjecture gives us the algebraic analysis realization of the half-transfer matrix of our (n,n) RSOS model. When $m > n$, it gives us a realization of the (m,n) impurity operator, i.e. of the operator made up from a half-infinite tower of (m,n) weights inserted into our (n,n) RSOS model.

4. Perturbation theory

In this section, we present the results of a perturbation theory check around $q = 0$ of our conjecture (3.9). We fix the values $(k,n) = (3,1)$ and check (3.9) for $m = 1$ and for $m = 2$. $(k,n) = (3,1)$ is the simplest model for which both the half-transfer matrix and the $m = n+1$ impurity operator are non-trivial. The perturbation theory analysis involves three main steps. Step 1 is an extension of the analysis of the $k = 2$ case carried out in [3].

Step 1. First of all, we compute a perturbative expansion for $|\mathrm{vac}\rangle \in \mathcal{H}^{2\Lambda_0,\Lambda_0;3\Lambda_0}$. This vector is defined to be the minimum eigenvalue eigenvector of the corner transfer matrix Hamiltonian H_{CTM}. H_{CTM} is in turn defined by

$$H_{\mathrm{CTM}} = -\left.\frac{\mathrm{d}A_{\mathrm{NW}}^{(2\Lambda_0,\Lambda_0;3\Lambda_0)}(\zeta)}{\mathrm{d}\zeta}\right|_{\zeta=1}$$

where $A_{\mathrm{NW}}^{(2\Lambda_0,\Lambda_0;3\Lambda_0)}(\zeta)$ is the corner transfer matrix of the (1, 1) RSOS model with $k = 3$.

We will use the following abbreviated notation for (m, n) Boltzmann weights:

$$W_k^{(m,n)}\left(\begin{array}{cc} a & b \\ c & d \end{array}\middle|\, \zeta\right) \equiv W_k^{(m,n)}\left(\begin{array}{cc} \lambda_a^{(k)} & \lambda_b^{(k)} \\ \lambda_c^{(k)} & \lambda_d^{(k)} \end{array}\middle|\, \zeta\right)$$

and we define $\overline{W}_k^{(1,1)}$ by

$$W_k^{(1,1)}\left(\begin{array}{cc} a & b \\ c & d \end{array}\middle|\, \zeta\right) = \frac{1}{\kappa^{(1,1)}(\zeta)}\frac{\eta(\zeta^2)}{\eta(\zeta^{-2})}\overline{W}_k^{(1,1)}\left(\begin{array}{cc} a & b \\ c & d \end{array}\middle|\, \zeta\right)$$

where $\kappa^{(1,1)}(\zeta)$ and $\eta(\zeta)$ are given by (2.2) and (A.11).

Let us write out the weights for the (1,1) RSOS model (these come from formulae (A.8), (A.9)). We have

$$\bar{\alpha}_k(\zeta) \equiv \overline{W}_k^{(1,1)}\left(\begin{array}{cc} a & a\pm 1 \\ a\pm 1 & a\pm 2 \end{array}\middle|\, \zeta\right) = 1 \tag{4.1}$$

$$\bar{\beta}_k^{a\pm}(\zeta) \equiv \overline{W}_k^{(1,1)}\left(\begin{array}{cc} a & a\pm 1 \\ a\mp 1 & a \end{array}\middle|\, \zeta\right) = q\frac{\Gamma_p(r_\mp)\Gamma_p(r_\mp)}{\Gamma_p(2s+r_\mp)\Gamma_p(-2s+r_\mp)}\frac{\Theta_p(\zeta^2)}{\Theta(q^2\zeta^2)} \tag{4.2}$$

$$\bar{\gamma}_k^{a\pm}(\zeta) \equiv \overline{W}_k^{(1,1)}\left(\begin{array}{cc} a & a\pm 1 \\ a\pm 1 & a \end{array}\middle|\, \zeta\right) = \zeta\frac{\Theta_p(q^2)\Theta_p(p^{r_\pm}\zeta^2)}{\Theta_p(q^2\zeta^2)\Theta_p(p^{r_\pm})} \tag{4.3}$$

where $r_- = 2(a+1)s$ and $r_+ = 1 - r_-$. Γ_p and Θ_p are defined in equation (A.10). The largest weight in our specified region $0 < -q < \zeta^{-1} < 1$ is $\bar{\gamma}_k^{a\pm}(\zeta)$.

Noting that $\bar{\alpha}_k(1) = 1$, $\bar{\beta}_k^{a\pm}(1) = 0$ and $\bar{\gamma}_k^{a\pm}(1) = 1$, our 'renormalized' corner transfer matrix Hamiltonian is given by

$$H_{\mathrm{CTM}}^r = R - \sum_{\ell=1}^{\infty} \ell \cdot O_\ell. \tag{4.4}$$

The operator O_ℓ acts as the identity on a path $|p\rangle \in \mathcal{P}^{2\Lambda_0,\Lambda_0;3\Lambda_0}$ everywhere except on the triple $(p(\ell+2), p(\ell+1), p(\ell))$, where its action is given by

$$O_\ell(a\pm 2, a\pm 1, a) = 0$$
$$O_\ell(a, a\pm 1, a) = \check{b}^{a\pm}(a, a\mp 1, a) + \check{c}^{a\pm}(a, a\pm 1, a)$$

with

$$\check{b}^{a\pm} \equiv \left.\frac{\mathrm{d}\bar{\beta}_3^{a\pm}(\zeta)}{\mathrm{d}\zeta}\right|_{\zeta=1} \qquad \check{c}^{a\pm} \equiv \left.\frac{\mathrm{d}\bar{\gamma}_3^{a\pm}(\zeta)}{\mathrm{d}\zeta}\right|_{\zeta=1}.$$

Here, and elsewhere in this section, we use the abbreviated notation a to indicate the weight $\lambda_a^{(k)}$.

Before giving the definition of the constant R, which fixes what we mean by renormalized, let us introduce some notation for certain paths $|p\rangle \in \mathcal{P}_{2\Lambda_0,\Lambda_0;3\Lambda_0}$. We use the notation $|\emptyset\rangle$ to indicate the ground-state path $|p_{2\Lambda_0,\Lambda_0}\rangle = (\cdots\cdots 1010)$. Then $|2\ell+1\rangle$, with $\ell > 0$, will indicate a path which differs from $|\emptyset\rangle$ only in that $p(2\ell+1) = 2$. Similarly,

$|2\ell_1+1, 2\ell_2+1, \ldots, 2\ell_M\rangle$ denotes a path that is the same as $|\emptyset\rangle$ except that $p(2\ell_1+1) = 2$, $p(2\ell_2+1) = 2, \ldots, p(2\ell_M+1) = 2$. Finally, $|2\ell+3, 2\ell+2, 2\ell+1\rangle$ indicates a path for which $p(2\ell+3) = 2$, $p(2\ell+2) = 3$ and $p(2\ell+1) = 2$. In steps 2 and 3, we will use a very similar notation for paths in other path spaces—but we will try to avoid confusion by always specifying which path space we are dealing with.

Now we come back to the meaning of (4.4). $R = \sum_{\ell=1}^{\infty} \ell \cdot R_\ell \mathrm{id}$ is fixed by the requirements

$$H^r_{\mathrm{CTM}}|\mathrm{vac}\rangle = 0 \tag{4.5}$$

$$\langle\emptyset|\mathrm{vac}\rangle = 1. \tag{4.6}$$

The r superscript on H^r_{CTM} indicates this choice of (re)normalization. The conditions (4.5) and (4.6) fix R_ℓ to be

$$R_{2\ell-1} = \check{c}^{0+} \qquad R_{2\ell} = (\check{c}^{1-} + \check{b}^{1+}\langle 2\ell+1|\mathrm{vac}\rangle).$$

It remains only to solve $H^r_{\mathrm{CTM}}|\mathrm{vac}\rangle = 0$ perturbatively by expanding both H^r_{CTM} and $|\mathrm{vac}\rangle$ around $q = 0$. We find

$$\begin{aligned}
|\mathrm{vac}\rangle = |\emptyset\rangle - q\sum_{\ell}|2\ell+1\rangle + q^2\Bigg(&\sum_{\ell_1\gg\ell_2}|2\ell_1+1, 2\ell_2+1\rangle + 2\sum_{\ell}|2\ell+3, 2\ell+1\rangle\Bigg)\\
&+q^3\Bigg(2\sum_{\ell}|2\ell+1\rangle - \sum_{\ell_1\gg\ell_2\gg\ell_3}|2\ell_1+1, 2\ell_2+1, 2\ell_3+1\rangle\\
&-2\sum_{\ell_1\gg\ell_2+1}|2\ell_1+1, 2\ell_2+3, 2\ell_2+1\rangle - 2\sum_{\ell_2\gg\ell_1}|2\ell_2+3, 2\ell_2+1, 2\ell_1+1\rangle\\
&-5\sum_{\ell}|2\ell+5, 2\ell+3, 2\ell+1\rangle - \sum_{\ell}|2\ell+3, 2\ell+2, 2\ell+1\rangle\Bigg) + \mathrm{O}(q^4)
\end{aligned} \tag{4.7}$$

where $\ell_1 \gg \ell_2$ means $\ell_1 > \ell_2 + 1$.

Step 2. In this step, we will compute $\iota(|v_{2\Lambda_0}\otimes v_{\Lambda_0}\rangle)$, $\iota(X^1_0(\zeta)|v_{2\Lambda_0}\otimes v_{\Lambda_0}\rangle)$ and $\iota(Z^{12}_{00;2}(\zeta)|v_{2\Lambda_0}\otimes v_{\Lambda_0}\rangle)$ perturbatively. $X^1_0(\zeta)$ and $Z^{12}_{00;2}(\zeta)$ are defined by (2.11) and (2.12), and ι is defined by (3.5)–(3.7) (recall that we are selectively indicating the weight $\lambda^{(3)}_a$ by the integer a).

To find $\iota : \Omega_{\xi,\eta,\lambda} \to \mathcal{H}_{\xi,\eta,\lambda}$, we must calculate the perturbative action of $X^{\lambda'}_{\lambda}(\zeta) : \Omega_{\xi,\eta;\lambda} \to \Omega_{\xi,\sigma(\eta);\lambda'}$. To do this, it is useful if we make the identification

$$\Omega_{\xi,\eta;\lambda} = \mathrm{Hom}_{U_q(\widehat{\mathfrak{sl}}_2)}(V(\lambda), V(\xi)\otimes V(\eta)).$$

Then for $\alpha \in \mathrm{Hom}_{U_q(\widehat{\mathfrak{sl}}_2)}(V(\lambda), V(\xi)\otimes V(\eta))$, $X^{\lambda'}_{\lambda}(\zeta)(\alpha)$ is defined via the commutative diagram

$$\begin{array}{ccc}
V(\xi)\otimes V(\eta) & \xrightarrow{\;1\otimes\Phi^{\sigma(\eta)V^{(1)}}_{\eta}(\zeta)\;} & V(\xi)\otimes V(\sigma(\eta)\otimes V^{(1)}_{\zeta}\\
\alpha\Big\uparrow & & \Big\uparrow X^{\lambda'}_{\lambda}(\zeta)(\alpha)\\
V(\lambda) & \xrightarrow{\;\Phi^{\lambda' V^{(1)}}_{\lambda}(\zeta)\;} & \sum_{\lambda'} V(\lambda')\otimes V^{(1)}_{\zeta}
\end{array}$$

Let us list the first few highest-weight elements of the various $V(\xi)\otimes V(\eta)$ that we shall need in this section. Note that if w is such a highest-weight element, then we have the identification $w=\alpha(v_\lambda)$.

In $V(2\Lambda_0)\otimes V(\Lambda_0)$, we have

$$x_1^{(0)} = v_{2\Lambda_0}\otimes v_{\Lambda_0}$$
$$x_1^{(2)} = v_{2\Lambda_0}\otimes f_0 v_{\Lambda_0} - q^2\frac{1}{[2]} f_0 v_{2\Lambda_0}\otimes v_{\Lambda_0}$$
$$x_2^{(2)} = \frac{1}{[2]} v_{2\Lambda_0}\otimes f_0 f_1 f_0 v_{\Lambda_0} - \frac{q^2}{[2]^2} f_0 v_{2\Lambda_0}\otimes f_1 f_0 v_{\Lambda_0} + \frac{q^4}{[2]^2} f_1 f_0 v_{2\Lambda_0}\otimes f_0 v_{\Lambda_0} + q^6\frac{1}{[2]^2([4]-[2])}(f_1 f_0^2 + (1-[3]) f_0 f_1 f_0) v_{2\Lambda_0}\otimes v_{\Lambda_0}.$$

In $V(2\Lambda_0)\otimes V(\Lambda_1)$:

$$x_1^{(1)} = v_{2\Lambda_0}\otimes v_{\Lambda_1}$$
$$x_2^{(1)} = \frac{1}{[2]} v_{2\Lambda_0}\otimes f_0 f_1 v_{\Lambda_1} - \frac{q^2}{[2]} f_0 v_{2\Lambda_0}\otimes f_1 v_{\Lambda_1} + \frac{q^4}{[2]^2} f_1 f_0 v_{2\Lambda_0}\otimes v_{\Lambda_1}$$
$$x_1^{(3)} = \frac{1}{[2]} v_{2\Lambda_0}\otimes f_0^2 f_1 v_{\Lambda_1} - \frac{q^2}{[2]} f_0 v_{2\Lambda_0}\otimes f_0 f_1 v_{\Lambda_1} + \frac{q^2}{[2]} f_0^2 v_{2\Lambda_0}\otimes f_1 v_{\Lambda_1} + \frac{q^6}{[2]([4]-[2])}(f_0 f_1 f_0 - f_1 f_0^2) v_{2\Lambda_0}\otimes v_{\Lambda_1}.$$

In $V(\Lambda_1+\Lambda_0)\otimes V(\Lambda_0)$:

$$y_1^{(1)} = v_{\Lambda_1+\Lambda_0}\otimes v_{\Lambda_0}$$
$$y_1^{(3)} = v_{\Lambda_1+\Lambda_0}\otimes f_0 v_{\Lambda_0} - q f_0 v_{\Lambda_1+\Lambda_0}\otimes v_{\Lambda_0}$$
$$y_2^{(1)} = \frac{1}{[2]} v_{\Lambda_1+\Lambda_0}\otimes f_1 f_0 v_{\Lambda_0} - q f_1 v_{\Lambda_1+\Lambda_0}\otimes f_0 v_{\Lambda_0} + \frac{q^4}{1-[3]^2}(f_1 f_0 - [3] f_0 f_1) v_{\Lambda_1+\Lambda_0}\otimes v_{\Lambda_0}$$
$$y_2^{(3)} = \frac{1}{[2]} v_{\Lambda_1+\Lambda_0}\otimes f_0 f_1 f_0 v_{\Lambda_0} - \frac{q}{[2]} f_0 v_{\Lambda_1+\Lambda_0}\otimes f_1 f_0 v_{\Lambda_0} + \frac{q^4}{1-[3]^2}(f_0 f_1 - [3] f_1 f_0) v_{\Lambda_1+\Lambda_0}\otimes f_0 v_{\Lambda_0} - \frac{q^5}{1-[3]^2}(f_0^2 f_1 - [3] f_0 f_1 f_0) v_{\Lambda_1+\Lambda_0}\otimes v_{\Lambda_0}.$$

In $V(\Lambda_1+\Lambda_0)\otimes V(\Lambda_1)$:

$$y_1^{(2)} = \overline{y_1^{(1)}} \qquad y_1^{(0)} = \overline{y_1^{(3)}} \qquad y_2^{(2)} = \overline{y_2^{(1)}} \qquad y_2^{(0)} = \overline{y_2^{(3)}}$$

where the bar operation exchanges 0 and 1 indices, e.g. $\overline{y_1^{(3)}} = v_{\Lambda_1+\Lambda_0}\otimes f_1 v_{\Lambda_1} - q f_1 v_{\Lambda_1+\Lambda_0}\otimes v_{\Lambda_1}$. The notation is such that $x_i^{(a)}\in\Omega_{2\Lambda_0,\Lambda_j;\lambda_a^{(3)}}$ and $y_i^{(a)}\in\Omega_{\Lambda_1+\Lambda_0,\Lambda_j;\lambda_a^{(3)}}$ (with $j\in\{0,1\}$).

We can then calculate the perturbative action of $X_\lambda^{\lambda'}(\zeta)$ on these vectors by making use of the perturbative action of $\Phi_{\eta}^{\sigma(\eta)V^{(1)}}(\zeta)$ and $\Phi_{\lambda}^{\lambda' V^{(1)}}(\zeta)$ given in appendix C. We find

$$\begin{aligned}
&X_0^1(\zeta)(x_1^{(0)}) = x_1^{(1)} + \zeta^2 q^3 x_2^{(1)} + \cdots \\
&X_1^0(\zeta)(x_1^{(1)}) = x_1^{(0)} + \cdots \qquad X_1^0(\zeta)(x_2^{(1)}) = \zeta^{-2}(q-q^3)x_1^{(0)} + \cdots \\
&X_1^2(\zeta)(x_1^{(1)}) = -\zeta q x_1^{(2)} - \zeta^3 q^4 x_2^{(2)} + \cdots \\
&X_1^2(\zeta)(x_2^{(1)}) = \zeta^{-1}(1-q^2)x_1^{(2)} - \zeta(q-q^3)x_2^{(2)} + \cdots \\
&X_2^1(\zeta)(x_1^{(2)}) = \zeta^{-1}x_1^{(1)} - q\zeta x_2^{(1)} + \cdots \\
&X_2^1(\zeta)(x_2^{(2)}) = \zeta^{-3}(q-q^3)x_1^{(1)} + \zeta^{-1}(1-q^2)x_2^{(1)} + \cdots \\
&X_2^3(\zeta)(x_1^{(2)}) = \zeta^2 q^2 x_1^{(3)} + \cdots \qquad X_2^3(\zeta)(x_2^{(2)}) = -(q-2q^3)x_1^{(3)} + \cdots \\
&X_3^2(\zeta)(x_1^{(3)}) = \zeta^{-2}x_1^{(2)} - q x_2^{(2)} + \cdots
\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}
&X_0^1(\zeta)(y_1^{(0)}) = \frac{1}{\zeta}y_1^{(1)} - \zeta q y_2^{(1)} + \cdots \\
&X_0^1(\zeta)(y_2^{(0)}) = \frac{1}{\zeta^3}(q-q^3)y_1^{(1)} + \frac{1}{\zeta}(1-q^2)y_2^{(1)} + \cdots \\
&X_1^0(\zeta)(y_1^{(1)}) = \zeta(-q+q^3)y_1^{(0)} + \cdots \\
&X_1^0(\zeta)(y_2^{(1)}) = \frac{1}{\zeta}(1-q^2)y_1^{(0)} + \zeta(-q+q^3)y_2^{(0)} + \cdots \\
&X_1^2(\zeta)(y_1^{(1)}) = y_1^{(2)} + \zeta^2 q^3 y_2^{(2)} + \cdots \\
&X_1^2(\zeta)(y_2^{(1)}) = \frac{1}{\zeta^2}(q-q^3)y_1^{(2)} - q^2 y_2^{(2)} + \cdots \\
&X_2^1(\zeta)(y_1^{(2)}) = y_1^{(1)} + \zeta^2 q^3 y_2^{(1)} + \cdots \\
&X_2^1(\zeta)(y_2^{(2)}) = \frac{1}{\zeta^2}(q-q^3)y_1^{(1)} - q^2 y_2^{(1)} + \cdots \\
&X_2^3(\zeta)(y_1^{(2)}) = \zeta(-q+q^3)y_1^{(3)} + \cdots \\
&X_2^3(\zeta)(y_2^{(2)}) = \frac{1}{\zeta}(1-q^2)y_1^{(3)} + \zeta(-q+q^3)y_2^{(3)} + \cdots \\
&X_3^2(\zeta)(y_1^{(3)}) = \frac{1}{\zeta}y_1^{(2)} - \zeta q y_2^{(2)} + \cdots \\
&X_3^2(\zeta)(y_2^{(3)}) = \frac{1}{\zeta^3}(q-q^3)y_1^{(2)} + \frac{1}{\zeta}(1-q^2)y_2^{(2)} + \cdots
\end{aligned} \tag{4.9}$$

where each of the coefficients is given to order q^3.

Let us go through the example of how to compute $X_0^1(\zeta)(x_1^{(0)})$ (or rather $X_0^1(\zeta)(\alpha_1^{(0)})$, where $x_i^{(a)} = \alpha_i^{(a)}(v_{\lambda_a^{(3)}})$—the x and y appearing in (4.8) and (4.9) refer in this context to the associated homomorphisms). First of all, it follows from (C.1) that we have

$$\begin{aligned}
(1\otimes\Phi_{\Lambda_0}^{\Lambda_1}(\zeta))x_1^{(0)} = v_{2\Lambda_0}\otimes\bigg(&v_{\Lambda_1}\otimes u_1 - q f_1 v_{\Lambda_1}\otimes u_0\zeta + q^3\frac{1}{[2]}f_0 f_1 v_{\Lambda_1}\otimes u_1\zeta^2 \\
&-q^4\frac{1}{[2]}f_1 f_0 f_1 v_{\Lambda_1}\otimes u_0\zeta^3 + \cdots\bigg).
\end{aligned} \tag{4.10}$$

Then, we use the perturbative expression for $\Phi_{3\Lambda_0}^{2\Lambda_0+\Lambda_1 V^{(1)}}(\zeta)v_{3\Lambda_0}$, given in equation (C.3) of appendix C, from which it follows that

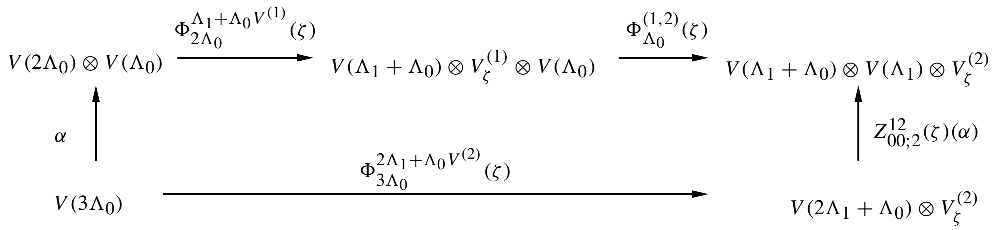
$$\begin{aligned}
(\alpha_i^{(1)}\otimes 1)\Phi_{3\Lambda_0}^{2\Lambda_0+\Lambda_1 V^{(1)}}(\zeta)v_{3\Lambda_0} = {}& x_i^{(1)}\otimes u_1 - q f_1 x_i^{(1)}\otimes u_0\zeta \\
&+q^5\frac{1}{[4]+[6]}([3]f_0 f_1 - f_1 f_0)x_i^{(1)}\otimes u_1 + \cdots.
\end{aligned} \tag{4.11}$$

Finally, we compute the coefficients $c_i(\zeta)$ in the expansion $X_0^1(\zeta)(\alpha_1^{(0)}) = \sum_i c_i(\zeta)\alpha_i^{(1)}$, by substituting the right-hand sides of (4.10) and (4.11) into the defining equation

$$(1 \otimes \Phi_{\Lambda_0}^{\Lambda_1}(\zeta))x_1^{(0)} = \sum_i c_i(\zeta)(\alpha_i^{(1)} \otimes 1)\Phi_{3\Lambda_0}^{2\Lambda_0+\Lambda_1 V^{(1)}}(\zeta)v_{3\Lambda_0}. \tag{4.12}$$

We find $c_1(\zeta) = 1$ and $c_2(\zeta) = \zeta^2 q^3$ to order q^3. These are the coefficients given in the first line of (4.8).

In a similar way, we can compute the action of $Z_{00;2}^{12}(\zeta)$, which is defined by (2.12), i.e. through the commutative diagram,



Making use of equations (C.2)–(C.8), we find

$$Z_{00;2}^{12}(\zeta)\big(x_1^{(0)}\big) = y_1^{(2)} + \zeta^2 \frac{[2]}{[4][3] - [2]} y_2^{(2)} + \cdots. \tag{4.13}$$

It remains to compute $\iota(x_1^{(0)})$, $\iota(X_0^1(\zeta)(x_1^{(0)}))$ and $\iota(Z_{00;2}^{12}(\zeta)(x_1^{(0)}))$. Let us go through the example of $\iota(x_1^{(0)})$. We must calculate the the path coefficients $c(p, x_1^{(0)})$ defined in (3.6) and (3.7). As an example, let us do this for the path $|3\rangle \in P_{2\Lambda_0,\Lambda_0;3\Lambda_0}$. First, using (4.8), we calculate the denominator $c^\ell(|\emptyset\rangle, x_1^{(0)})$ of (3.6) for several values of ℓ. In fact $c^\ell(|\emptyset\rangle, x_1^{(0)}) = 1 + \mathrm{O}(q^4)$ for all ℓ, and so, since we are computing only up to order q^3, it never enters the ratio (3.6). We find the numerator $c^\ell(|3\rangle, x_1^{(0)})$ has the following values:

$$\begin{aligned}
&c^4(|3\rangle, x_1^{(0)}) = \langle x_1^0 | X_1^0(1) X_2^1(1) X_1^2(1) X_0^1(1) | x_1^{(0)}\rangle = -q + 2q^3 + \mathrm{O}(q^5)\\
&c^5(|3\rangle, x_1^{(0)}) = \langle x_1^{(1)} | X_0^1(1) X_1^0(1) X_2^1(1) X_1^2(1) X_0^1(1) | x_1^{(0)}\rangle = -q + 2q^3 + \mathrm{O}(q^5)\\
&\vdots\\
&c^\ell(|3\rangle, x_1^{(0)}) = -q + 2q^3 + \mathrm{O}(q^5).
\end{aligned} \tag{4.14}$$

Hence, from (3.6), we have $c(|3\rangle, x_1^{(0)}) = -q + 2q^3 + \mathrm{O}(q^4)$. $c(p, x_1^{(0)})$ of any path $|p\rangle \in \mathcal{P}_{2\Lambda_0,\Lambda_0;3\Lambda_0}$ can be calculated in a similar way. We computed the coefficients of a range of example paths in $\iota(x_1^{(0)})$ to order q^3 (to be precise we considered the paths $|\emptyset\rangle$, $|3\rangle$, $|5\rangle$, $|7\rangle$, $|7,5\rangle$, $|9,3\rangle$, $|9,5\rangle$, $|11,5\rangle$, $|7,5,3\rangle$, $|9,5,3\rangle$, $|11,5,3\rangle$ and $|5,4,3\rangle$). We found that the coefficients of each of these paths were equal to those in expression (4.7) for $|\mathrm{vac}\rangle$, so our perturbative results are consistent with the identification $\iota(x_1^{(0)}) = |\mathrm{vac}\rangle$.

In a similar way we have computed the coefficients of certain paths in $\mathcal{P}_{0,1;1}$ contributing to $\iota(X_0^1(\zeta)(x_1^{(0)}))$. The notation for paths in $\mathcal{P}_{0,1;1}$ is such that $|\emptyset\rangle = (\cdots 010101)$, and $|2\ell\rangle$ differs from $|\emptyset\rangle$ only in that $p(2\ell) = 2$. Listing the path in $\mathcal{P}_{0,1;1}$ and then the coefficient $c(p, X_0^1(\zeta)(x_1^{(0)}))$, we have to order q^3

$$\begin{array}{ll}
|\emptyset\rangle & 1\\
|2\rangle & -q + (1+\zeta^2)q^3\\
|2\ell\rangle_{\ell>1} & -q + 2q^3.
\end{array} \tag{4.15}$$

Finally, we have computed to order q^3 the coefficients for certain paths in $\mathcal{P}_{11;2}$ contributing to $\iota(Z^{12}_{00;2}(\zeta)|x^{(0)}_1\rangle)$. Here the path notation is $|\emptyset\rangle = (\cdots 121212)$, $|2\ell+1\rangle$ differs from it only in that $p(2\ell+1) = 0$ and $|2\ell\rangle$ differs from it only in that $p(2\ell) = 3$. Listing the path and then the coefficient $c(p, Z^{12}_{00;2}(\zeta)(x^{(0)}_1))$, we have

$$\begin{array}{ll} |\emptyset\rangle & 1 \\ |2\rangle & -q+2q^3 \\ |2\ell+1\rangle_{\ell>0} & -q+3q^3 \\ |2\ell\rangle_{\ell>1} & -q+3q^3. \end{array} \tag{4.16}$$

Step 3. In this step, we carry out a lattice perturbation theory calculation of ${}_NX^1_0(\zeta)\circ\rho_N|\text{vac}\rangle$ and ${}_NZ^{12}_{00;2}(\zeta)\circ\rho_N\circ|\text{vac}\rangle$. We compare with the results of step 2 and hence check the conjecture (3.9).

First we shall calculate the action of ${}_NX^1_0(\zeta)$ on $\rho_N|\text{vac}\rangle$, where ${}_NX^1_0(\zeta)$ is defined to be the lattice operator (3.8) in the case when $(m,n)=(1,1)$. Define α, $\beta^{a\pm}$ and $\gamma^{a\pm}$ to be a factor of $\eta(\zeta^2)/(k^{(1,1)}(\zeta)\eta(\zeta^{-2}))$ times $\bar{\alpha}_3(\zeta)$, $\bar{\beta}^{a\pm}_3(\zeta)$ and $\bar{\gamma}^{a\pm}_3(\zeta)$ respectively. Then, as a series in q, we have $\alpha(\zeta)=\mathrm{O}(1)$, $\beta^{a\pm}(\zeta)=\mathrm{O}(q)$, $\gamma^{a\pm}(\zeta)=\mathrm{O}(1)$.

Let us compute the coefficients of $|\emptyset\rangle_N$ and $|2\ell\rangle_N$ in ${}_NX^1_0(\zeta)\circ\rho_N|\text{vac}\rangle$ (where $|p\rangle_N = \rho_N|p\rangle$, and $|\emptyset\rangle$ and $|2\ell\rangle$ are as defined above (4.15)). We introduce the notation $\gamma = (\gamma^{0+}\gamma^{1-})^{1/2}$ and define $f^{(1,1)}_N(\zeta,q)$ by

$$f^{(1,1)}_N(\zeta,q) = \begin{cases} 1+(-1+\zeta^{-2})q^2+\mathrm{O}(q^4) & \text{for } N \text{ even} \\ 1+\mathrm{O}(q^4) & \text{for } N \text{ odd.} \end{cases} \tag{4.17}$$

Then, the coefficients of $|\emptyset\rangle_N$, $|2\rangle_N$ and $|2\ell\rangle_N$ ($\ell>1$) in ${}_NX^1_0(\zeta)\circ\rho_N|\text{vac}\rangle$ when N is large and even are given to order q^3 by

$$\begin{aligned} &\gamma^N - q\alpha\beta^{1+}\gamma^{N-2}(N-2)/2 = f^{(1,1)}_N(\zeta,q) \\ &\alpha\beta^{1-}\gamma^{N-2} + (-q+2q^3)\alpha^2\gamma^{1+}\gamma^{1-}\gamma^{N-4} - q\alpha^2\beta^{1+}\beta^{(1-)}\gamma^{N-4}(N-4)/2 \\ &\qquad + q^2\alpha^3\gamma^{1+}\gamma^{1-}\beta^{1+}\gamma^{N-6}(N-2)/2 = f^{(1,1)}_N(\zeta,q)(-q+(1+\zeta^2)q^3) \qquad \text{and} \\ &\alpha\beta^{1-}\gamma^{N-2} + (-q+2q^3)\alpha^2\gamma^{1+}\gamma^{1-}\gamma^{N-4} + (-q+2q^3)\beta^{1-}\beta^{1+}\gamma^{2-}\gamma^{0+}\gamma^{N-4} \\ &\qquad + (-q+2q^3)\alpha^2\beta^{1+}\beta^{1-}\gamma^{N-4}(N-6)/2 + q^2\alpha^3\beta^{1+}\gamma^{1+}\gamma^{1-}\gamma^{N-6}(N-4)/2 \\ &\qquad + 2q^2\alpha\beta^{1+}\gamma^{2-}\gamma^{1+}\gamma^{N-4} = f^{(1,1)}_N(\zeta,q)(-q+2q^3) \end{aligned}$$

respectively. When N is large and odd, the three coefficients are

$$\begin{aligned} &\gamma^{0+}\gamma^{N-1} - q\alpha\beta^{1+}\gamma^{0+}\gamma^{N-3}(N-1)/2 = f^{(1,1)}_N(\zeta,q) \\ &\alpha\beta^{1-}\gamma^{0+}\gamma^{N-3} + (-q+2q^3)\alpha^2\gamma^{1+}\gamma^{N-3} - q\alpha^2\beta^{1+}\beta^{1-}\gamma^{0+}\gamma^{N-5}(N-3)/2 \\ &\qquad + q^2\alpha^3\gamma^{1+}\beta^{1+}\gamma^{N-5}(N-1)/2 = f^{(1,1)}_N(\zeta,q)(-q+(1+\zeta^2)q^3) \qquad \text{and} \\ &\alpha\beta^{1-}\gamma^{0+}\gamma^{N-3} + (-q+2q^3)\alpha^2\gamma^{1+}\gamma^{N-3} + (-q+2q^3)\beta^{1-}\beta^{1+}\gamma^{2-}(\gamma^{0+})^2\gamma^{N-5} \\ &\qquad + (-q+2q^3)\alpha^2\beta^{1+}\beta^{1-}\gamma^{0+}\gamma^{N-5}(N-5)/2 + q^2\alpha^3\beta^{1+}\gamma^{1+}\gamma^{N-5}(N-3)/2 \\ &\qquad + 2q^2\alpha\beta^{1+}\gamma^{2-}\gamma^{1+}\gamma^{0+}\gamma^{N-5} = f^{(1,1)}_N(\zeta,q)(-q+2q^3). \end{aligned}$$

Comparing these coefficients with those of (4.15), we see that our perturbation theory calculation is consistent with conjecture (3.9) in the case $(m,n)=(1,1)$.

In order to consider $Z^{12}_{00;2}(\zeta)\circ\rho_N\circ|\text{vac}\rangle$ we must first introduce some notation for Boltzmann weights. There are six independent Boltzmann weights, the formulae for which are given by (A.8), (A.9).

We denote them by

$$\begin{aligned}
A \;&= W_3^{(2,1)}\left(\begin{array}{cc} 0 & 2 \\ 1 & 3 \end{array}\middle|\, \zeta\right) = W_3^{(2,1)}\left(\begin{array}{cc} 3 & 1 \\ 2 & 0 \end{array}\middle|\, \zeta\right) \\
B_{12}^{\mathrm{e}} &= W_3^{(2,1)}\left(\begin{array}{cc} 1 & 1 \\ 2 & 2 \end{array}\middle|\, \zeta\right) = W_3^{(2,1)}\left(\begin{array}{cc} 2 & 2 \\ 1 & 1 \end{array}\middle|\, \zeta\right) \\
B_{12}^{\mathrm{d}} &= W_3^{(2,1)}\left(\begin{array}{cc} 1 & 3 \\ 0 & 2 \end{array}\middle|\, \zeta\right) = W_3^{(2,1)}\left(\begin{array}{cc} 2 & 0 \\ 3 & 1 \end{array}\middle|\, \zeta\right) \\
C_{10}^{\mathrm{e}} &= W_3^{(2,1)}\left(\begin{array}{cc} 1 & 1 \\ 2 & 0 \end{array}\middle|\, \zeta\right) = W_3^{(2,1)}\left(\begin{array}{cc} 2 & 2 \\ 1 & 3 \end{array}\middle|\, \zeta\right) \\
C_{12}^{\mathrm{e}} &= W_3^{(2,1)}\left(\begin{array}{cc} 1 & 1 \\ 0 & 2 \end{array}\middle|\, \zeta\right) = W_3^{(2,1)}\left(\begin{array}{cc} 2 & 2 \\ 3 & 1 \end{array}\middle|\, \zeta\right) \\
C_{01}^{\mathrm{d}} &= W_3^{(2,1)}\left(\begin{array}{cc} 0 & 2 \\ 1 & 1 \end{array}\middle|\, \zeta\right) = W_3^{(2,1)}\left(\begin{array}{cc} 3 & 1 \\ 2 & 2 \end{array}\middle|\, \zeta\right) \\
C_{12}^{\mathrm{d}} &= W_3^{(2,1)}\left(\begin{array}{cc} 1 & 3 \\ 2 & 2 \end{array}\middle|\, \zeta\right) = W_3^{(2,1)}\left(\begin{array}{cc} 2 & 0 \\ 1 & 1 \end{array}\middle|\, \zeta\right).
\end{aligned}$$

The notation is such that an e superscript implies that the NW and NE entries are equal, and a d superscript implies that they are different. The subscripts give the values of the (NW, SE) pair of entries (for one of the members of a pair of equal Boltzmann weights). B weights have zero or two horizontal pairs in which the entries are equal, C weights have one such pair. As q-series, the A and C weights are $\mathrm{O}(1)$ and the B weights are $\mathrm{O}(q)$. Now we compute the $|\emptyset\rangle_N$, $|2\ell\rangle_N$ and $|2\ell+1\rangle_N$ contributions to $Z_{00;2}^{12}(\zeta)\circ\rho_N\circ|\mathrm{vac}\rangle$ (where $|\emptyset\rangle$, $|2\ell\rangle$ and $|2\ell+1\rangle$ are as defined above (4.16)). Let us define $f_N^{(2,1)}(\zeta,q)$ by

$$f_N^{(2,1)}(\zeta,q) = \begin{cases} 1+q^3/\zeta^2+\mathrm{O}(q^4) & \text{for } N \text{ even} \\ 1+q^2/2+\mathrm{O}(q^4) & \text{for } N \text{ odd.} \end{cases} \tag{4.18}$$

Then, the respective coefficients of $|\emptyset\rangle_N$, $|2\rangle_N$, $|2\ell+1\rangle_N$ $(\ell > 0)$ and $|2\ell\rangle_N$ $(\ell > 1)$ in $Z_{00;2}^{12}(\zeta)\circ\rho_N\circ|\mathrm{vac}\rangle$ when N is large and even are given up to order q^3 by

$$\begin{aligned}
&C^N + (B_{12}^{\mathrm{e}})^2 C^{N-2}(-q)(N-2)/2 = f_N^{(2,1)}(\zeta,q) \\
&AB_{12}^{\mathrm{d}}C^{N-2} + AB_{12}^{\mathrm{e}}C_{12}^{\mathrm{e}}C_{12}^{\mathrm{d}}C^{N-4}(-q) = f_N^{(2,1)}(\zeta,q)(-q+2q^3) \\
&C_{01}^{\mathrm{d}}C_{12}^{\mathrm{e}}C^{N-2}(-q+2q^3) = f_N^{(2,1)}(\zeta,q)(-q+3q^3) \\
&AB_{12}^{\mathrm{d}}C^{N-2} + B_{12}^{\mathrm{d}}B_{12}^{\mathrm{e}}C_{01}^{\mathrm{d}}C_{10}^{\mathrm{e}}C^{N-4}(-q) + AB_{12}^{\mathrm{e}}C_{12}^{\mathrm{d}}C_{12}^{\mathrm{e}}C^{N-4}(-q) = f_N^{(2,1)}(\zeta,q)(-q+3q^3).
\end{aligned}$$

When N is large and odd, they are

$$\begin{aligned}
&C_{10}^{\mathrm{e}}C^{N-1} + B_{12}^{\mathrm{e}}B_{12}^{\mathrm{e}}C_{10}^{\mathrm{e}}C^{N-3}(-q)(N-1)/2 = f_N^{(2,1)}(\zeta,q) \\
&AB_{12}^{\mathrm{d}}C_{10}^{\mathrm{e}}C^{N-3} + AB_{12}^{\mathrm{e}}C_{12}^{\mathrm{e}}C^{N-3}(-q) = f_N^{(2,1)}(\zeta,q)(-q+2q^3) \\
&C_{01}^{\mathrm{d}}C_{12}^{\mathrm{e}}C_{10}^{\mathrm{e}}C^{N-3}(-q+2q^3) = f_N^{(2,1)}(\zeta,q)(-q+3q^3) \\
&AB_{12}^{\mathrm{d}}C_{10}^{\mathrm{e}}C^{N-3} + B_{12}^{\mathrm{d}}B_{12}^{\mathrm{e}}C_{01}^{\mathrm{d}}(C_{10}^{\mathrm{e}})^2C^{N-5}(-q) + AB_{12}^{\mathrm{e}}C_{12}^{\mathrm{e}}C^{N-3}(-q) \\
&\qquad = f_N^{(2,1)}(\zeta,q)(-q+3q^3).
\end{aligned}$$

Comparing these coefficients with those of (4.16), we see that our perturbation theory calculation is consistent with the conjecture (3.9) in the case $(m,n) = (2,1)$.

5. Discussion

We have constructed a realization of impurity operators within the algebraic analysis picture of RSOS models. It is now a straightforward step to extend the approach described in [10] in order to write down trace expressions for correlation functions of impurity insertions in these models. It should also be feasible to construct a free-field realization of our impurity operators within the scheme of [11], and to compute integral formulae for the correlation functions.

Suppose $n = 1$. Then if q were equal to 1, our definition (2.11), (2.12) of X and Z_m would coincide with the coset construction of the Virasoro q-primary fields $\Phi_{(1,2)}$ and $\Phi_{(m,m+1)}$ respectively. A q-Virasoro algebra was constructed in terms of a free-field realization in [12], and in terms of a q-coset realization in [9]. A definition of q-primary fields, or q-vertex operators, was given in [13] (see also [14]). We anticipate that our X and Z_m give a coset construction of the q-vertex operators which are deformations of $\Phi_{(1,2)}$ and $\Phi_{(m,m+1)}$.

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Appendix A. Commutation relations of $\Phi_\lambda^{\lambda' V^{(1)}}(\zeta_1)$ and $\Phi_\lambda^{\lambda' V^{(n)}}(\zeta_2)$

In this appendix, we solve the q-KZ equation in order to derive the commutation relations for $\Phi_\lambda^{\lambda' V^{(1)}}(\zeta_1)$ and $\Phi_\lambda^{\lambda' V^{(n)}}(\zeta_2)$. In this way, we arrive at the explicit expressions for the connection coefficients $C_k^{(n,1)}$ and $C_k^{(1,n)}$.

In order to formulate and solve the q-KZ equation it is convenient to work with a different evaluation module, namely the homogeneous evaluation module $(V_n)_z$ defined in [6] in terms of vectors $v_i^{(n)}$, $i \in \{0, 1, \ldots, n\}$ (this module is labelled as $V_z^{(n)}$ in [6]). The isomorphism between this and the principal evaluation module $V_\zeta^{(n)}$ used elsewhere in this paper is

$$C_n(\zeta) : V_\zeta^{(n)} \xrightarrow{\sim} (V_n)_z \qquad u_j^{(n)} \longmapsto c_j^{(n)} \zeta^j v_j^{(n)} \tag{A.1}$$

where $c_j^{(n)} = \begin{bmatrix} n \\ j \end{bmatrix}_q^{\frac{1}{2}} q^{\frac{j}{2}(n-j)}$, and we identify $\zeta^2 = z$. $\begin{bmatrix} a \\ b \end{bmatrix}_q$ is the standard q-binomial coefficient.

We define normalized homogeneous intertwiners

$$\begin{aligned} &\tilde{\Phi}_\lambda^{\mu V_n}(z) : V(\lambda) \longrightarrow V(\mu) \otimes (V_n)_z \\ &v_\lambda \longmapsto v_\mu + v_j^{(n)} + \cdots \qquad \text{where} \quad \lambda = \mu + (n-2j)\bar{\rho} \end{aligned} \tag{A.2}$$

exactly as in section 3.3 of [6]. The relation to the principal intertwiners defined in section 2.2 above is

$$\Phi_\lambda^{\mu V^{(n)}}(\zeta) = c_j^{(n)} \zeta^j C_n(\zeta)^{-1} \tilde{\Phi}_\lambda^{\mu V_n}(z = \zeta^2) \qquad \text{where} \quad \lambda = \mu + (n-2j)\bar{\rho}.$$

Now define the matrix element

$$\Psi^{(m,n)}(z_1/z_2) = \langle \nu | \tilde{\Phi}_\mu^{\nu V_m}(z_2) \tilde{\Phi}_\lambda^{\mu V_n}(z_1) | \lambda \rangle \in (V_m)_{z_2} \otimes (V_n)_{z_1}.$$

The q-KZ equation for $\Psi^{(m,n)}(z_1/z_2)$ is given by equations (A.18) and (A.19) of [6]. Let $\lambda = \lambda_a^{(k)}$, and define the function $\gamma(z)$ by

$$\gamma(z) = \frac{(pzq^{1-n}; p, q^4)_\infty (pzq^{3+n}; p, q^4)_\infty}{(pzq^{5+n}; p, q^4)_\infty (pzq^{-1-n}; p, q^4)_\infty}.$$

Then solving the q-KZ equation, we find the following.
When $\mu = \lambda_+$,

$$\Psi^{(n,1)}(z) = \gamma(z)\Bigg(\phi\begin{pmatrix} -2s(1+j) & 2s(a+1-n+j) \\ \multicolumn{2}{c}{2s(a+1)} \end{pmatrix}; zpq^{1+n}\Bigg) v_j^{(n)} \otimes v_1^{(1)}$$
$$+q^{2(a+1)+n-j}\frac{1-q^{2(j-n)}}{1-q^{2(a+1)}}$$
$$\times\phi\left(\begin{matrix} 1-2s(1+j) & 2s(a+1-n+j) \\ \multicolumn{2}{c}{1+2s(a+1)} \end{matrix}; zpq^{1+n}\right) v_{j+1}^{(n)} \otimes v_0^{(1)}\Bigg). \tag{A.3}$$

When $\mu = \lambda_-$,

$$\Psi^{(n,1)}(z) = \gamma(z)\Bigg(\phi\left(\begin{matrix} 2s(-n+j-1) & 1-2s(a+j+1) \\ \multicolumn{2}{c}{1-2s(a+1)} \end{matrix}; zpq^{1+n}\right) v_j^{(n)} \otimes v_0^{(1)}$$
$$+zpq^{-2(a+1)+j}\frac{1-q^{-2j}}{1-pq^{-2(a+1)}}$$
$$\times\phi\left(\begin{matrix} 1+2s(-n+j-1) & 1-2s(a+j+1) \\ \multicolumn{2}{c}{2-2s(a+1)} \end{matrix}; zpq^{1+n}\right) v_{j-1}^{(n)} \otimes v_1^{(1)}\Bigg). \tag{A.4}$$

When $\nu = \mu_+$,

$$\Psi^{(1,n)}(z) = \gamma(z)\Bigg(\phi\left(\begin{matrix} -2s(1+j) & 1-2s(a+j+2) \\ \multicolumn{2}{c}{1-2s(a-n+2j+2)} \end{matrix}; zpq^{1+n}\right) v_1^{(1)} \otimes v_j^{(n)}$$
$$+zq^{j-n}\frac{1-q^{2(n-j)}}{1-p^{-1}q^{2(a+2-n+2j)}}$$
$$\times\phi\left(\begin{matrix} 1-2s(1+j) & 1-2s(a+j+2) \\ \multicolumn{2}{c}{2-2s(a-n+2j+2)} \end{matrix}; zpq^{1+n}\right) v_0^{(1)} \otimes v_{j+1}^{(n)}\Bigg). \tag{A.5}$$

When $\nu = \mu_-$,

$$\Psi^{(1,n)}(z) = \gamma(z)\Bigg(\phi\left(\begin{matrix} 2s(-n+j-1) & 2s(a-n+j) \\ \multicolumn{2}{c}{2s(a-n+2j)} \end{matrix}; zpq^{1+n}\right) v_0^{(1)} \otimes v_j^{(n)}$$
$$+q^{-j}\frac{1-q^{2j}}{1-q^{-2(a-n+2j)}}$$
$$\times\phi\left(\begin{matrix} 1+2s(-n+j-1) & 2s(a-n+j) \\ \multicolumn{2}{c}{1+2s(a-n+2j)} \end{matrix}; zpq^{1+n}\right) v_1^{(1)} \otimes v_{j-1}^{(n)}\Bigg). \tag{A.6}$$

In all cases, j is determined uniquely by the requirement that weight $(\tilde{\Psi}(z)) = \lambda - \nu$. The function ϕ is the basic hypergeometric series

$$\phi\left(\begin{matrix} \alpha & \beta \\ \multicolumn{2}{c}{\gamma} \end{matrix}; z\right) = {}_2\phi_1\left(\begin{matrix} p^\alpha & p^\beta \\ \multicolumn{2}{c}{p^\gamma} \end{matrix}; p, z\right) = \sum_{n=0}^{\infty} \frac{(p^\alpha; p)_n (p^\beta; p)_n}{(p^\gamma; p)_n (p; p)_n} z^n.$$

The normalization of the first term in each of (A.3)–(A.6) is fixed by (A.2). The normalization of the second term follows from the q-KZ equation, and is computed by making use of the identities

$$(1-zp^\alpha)\phi\left(\begin{matrix} \alpha & \beta \\ \multicolumn{2}{c}{\gamma} \end{matrix}; pz\right) - (1-z)\phi\left(\begin{matrix} \alpha & \beta \\ \multicolumn{2}{c}{\gamma} \end{matrix}; z\right) = z(p^\beta - p^\gamma)\frac{(1-p^\alpha)}{(1-p^\gamma)}\phi\left(\begin{matrix} 1+\alpha & \beta \\ \multicolumn{2}{c}{1+\gamma} \end{matrix}; pz\right)$$

$$(1-zp^{\alpha+\beta+\gamma})\phi\left(\begin{matrix} \alpha & \beta \\ \multicolumn{2}{c}{\gamma} \end{matrix}; pz\right) - (1-zp^{\beta-\gamma})\phi\left(\begin{matrix} \alpha & \beta \\ \multicolumn{2}{c}{\gamma} \end{matrix}; z\right)$$
$$= -z(1-p^{\beta-\gamma})\frac{(1-p^\alpha)}{(1-p^\gamma)}\phi\left(\begin{matrix} 1+\alpha & \beta \\ \multicolumn{2}{c}{1+\gamma} \end{matrix}; z\right).$$

Given (A.3)–(A.6), the explicit form of the homogeneous R-matrix $\bar{R}^{(1,n)}(z)$ given in section 3.2 of [6], the connection formula (B.8) of [6] and the isomorphism (A.1), one can then compute the connection coefficients $C_k^{(n,1)}$ and $C_k^{(1,n)}$ defined in (2.6). We find

$$C_k^{(n,1)}\left(\begin{array}{cc} \lambda & \mu \\ \mu' & \nu \end{array}\middle|\zeta\right) = \frac{1}{\kappa^{(n,1)}(\zeta)}\bar{C}_k^{(n,1)}\left(\begin{array}{cc} \lambda & \mu \\ \mu' & \nu \end{array}\middle|\zeta\right) \tag{A.7}$$

where

$$\begin{aligned}
\bar{C}_k^{(n,1)}\left(\begin{array}{cc} \lambda & \nu_+ \\ \lambda_+ & \nu \end{array}\middle|\zeta\right) &= \zeta q^{\frac{1}{2}(n-2j+1)}\frac{[n-j+1]^{\frac{1}{2}}}{[j]^{\frac{1}{2}}}\frac{\eta(\zeta^2)}{\eta(\zeta^{-2})} \\
&\quad\times\frac{\Gamma_p(2s(a+2j-n))\Gamma_p(1-2s(a+1))\Theta_p(pq^{-2(a+j)+n-1}\zeta^2)}{\Gamma_p(1+2s(j-1-n))\Gamma_p(2sj)\Theta_p(q^{1+n}\zeta^2)} \\
\bar{C}_k^{(n,1)}\left(\begin{array}{cc} \lambda & \nu_+ \\ \lambda_- & \nu \end{array}\middle|\zeta\right) &= q^j\frac{\eta(\zeta^2)}{\eta(\zeta^{-2})} \\
&\quad\times\frac{\Gamma_p(2s(a+2j-n))\Gamma_p(2s(a+1))\Theta_p(q^{-2j+n+1}\zeta^2)}{\Gamma_p(2s(a+j-n))\Gamma_p(2s(a+j+1))\Theta_p(q^{1+n}\zeta^2)}
\end{aligned} \tag{A.8}$$

with j given by $\nu_+ + (n-2j)\bar{\rho} = \lambda$, and

$$\begin{aligned}
\bar{C}_k^{(n,1)}\left(\begin{array}{cc} \lambda & \nu_- \\ \lambda_+ & \nu \end{array}\middle|\zeta\right) &= q^{n-j}\frac{\eta(\zeta^2)}{\eta(\zeta^{-2})} \\
&\quad\times\frac{\Gamma_p(1-2s(a+2j-n+2))\Gamma_p(1-2s(a+1))\Theta_p(q^{2j+1-n}\zeta^2)}{\Gamma_p(1-2s(a+j+2))\Gamma_p(1-2s(a+j+1-n))\Theta_p(q^{1+n}\zeta^2)} \\
\bar{C}_k^{(n,1)}\left(\begin{array}{cc} \lambda & \nu_- \\ \lambda_- & \nu \end{array}\middle|\zeta\right) &= \zeta q^{\frac{1}{2}(2j-n+1)}\frac{[j+1]^{\frac{1}{2}}}{[n-j]^{\frac{1}{2}}}\frac{\eta(\zeta^2)}{\eta(\zeta^{-2})} \\
&\quad\times\frac{\Gamma_p(1-2s(a+2j-n+2))\Gamma_p(2s(a+1))\Theta_p(q^{2a+2j+3-n}\zeta^2)}{\Gamma_p(1-2s(1+j))\Gamma_p(2s(n-j))\Theta_p(q^{1+n}\zeta^2)}
\end{aligned} \tag{A.9}$$

with j given by $\nu_- + (n-2j)\bar{\rho} = \lambda$. The functions Γ_p and Θ_p are defined as usual by

$$\Gamma_p(z) = \frac{(p;p)_\infty}{(p^z;p)_\infty}(1-p)^{1-z} \qquad \Theta_p(z) = (p;p)_\infty(z;p)_\infty(pz^{-1};p)_\infty \tag{A.10}$$

and $\eta(\zeta)$ is defined by

$$\begin{aligned}
\eta(z) &= \frac{(pzq^{1+n};p,q^4)_\infty(pzq^{3-n};p,q^4)_\infty}{(pzq^{1-n};p,q^4)_\infty(pzq^{3+n};p,q^4)_\infty} \\
\text{with}\quad (a;b,c)_\infty &\equiv \prod_{n_1,n_2=0}^{\infty}(1-a\,b^{n_1}c^{n_2}).
\end{aligned} \tag{A.11}$$

We also find

$$C_k^{(1,n)}\left(\begin{array}{cc} \lambda & \mu \\ \mu' & \nu \end{array}\middle|\zeta\right) = C_k^{(n,1)}\left(\begin{array}{cc} \nu & \mu \\ \mu' & \lambda \end{array}\middle|\zeta\right) \tag{A.12}$$

such that the Boltzmann weights of section 3.1 are given by

$$W_k^{(n,1)}\left(\begin{array}{cc} \lambda & \mu \\ \mu' & \nu \end{array}\middle|\zeta\right) = C_k^{(n,1)}\left(\begin{array}{cc} \lambda & \mu \\ \mu' & \nu \end{array}\middle|\zeta\right).$$

Appendix B. Commutation relations of $\Phi_{\lambda}^{(n,n+k)}(\zeta)$

In this appendix we give a proof of the commutation relations

$$R^{(n+k,n+k)}(\zeta)\Phi_{\sigma(\lambda)}^{(n,n+k)}(\zeta_1)\Phi_{\lambda}^{(n,n+k)}(\zeta_2) = \Phi_{\sigma(\lambda)}^{(n,n+k)}(\zeta_2)\Phi_{\lambda}^{(n,n+k)}(\zeta_1)R^{(n,n)}(\zeta) \quad \text{(B.1)}$$

where $\zeta = \zeta_1/\zeta_2$. The proof will be inductive on the level k.

(B.1) is shown for $k = 1$ in [7], and we make the assumption that it is true for $k = \ell - 1$. Let $\lambda = \mu + \Lambda_i$ and consider

$$R^{(n+\ell,n+\ell)}(\zeta)(\Phi_{\Lambda_{1-i}}^{(n+\ell-1,n+\ell)}(\zeta_1)\Phi_{\sigma(\mu)}^{(n,n+\ell-1)}(\zeta_1))(\Phi_{\Lambda_i}^{(n+\ell-1,n+\ell)}(\zeta_2)\Phi_{\mu}^{(n,n+\ell-1)}(\zeta_2)) \quad \text{(B.2)}$$

which is an intertwiner $V_{\zeta_1}^{(n)} \otimes V_{\zeta_2}^{(n)} \otimes V(\mu) \otimes V(\Lambda_i) \to V(\mu) \otimes V(\Lambda_i) \otimes V_{\zeta_2}^{(n+\ell)} \otimes V_{\zeta_1}^{(n+\ell)}$. Since $\Phi_{\sigma(\mu)}^{(n,n+\ell-1)}(\zeta_1)$ and $\Phi_{\Lambda_i}^{(n+\ell-1,n+\ell)}(\zeta_2)$ act on different spaces, they commute. So (B.2) is equal to

$$R^{(n+\ell,n+\ell)}(\zeta)\Phi_{\Lambda_{1-i}}^{(n+\ell-1,n+\ell)}(\zeta_1)\Phi_{\Lambda_i}^{(n+\ell-1,n+\ell)}(\zeta_2)\Phi_{\sigma(\mu)}^{(n,n+\ell-1)}(\zeta_1)\Phi_{\mu}^{(n,n+\ell-1)}(\zeta_2).$$

Using (B.1) when $k = 1$, this is given by

$$\Phi_{\Lambda_{1-i}}^{(n+\ell-1,n+\ell)}(\zeta_2)\Phi_{\Lambda_i}^{(n+\ell-1,n+\ell)}(\zeta_1)R^{(n+\ell-1,n+\ell-1)}(\zeta)\Phi_{\sigma(\mu)}^{(n,n+\ell-1)}(\zeta_1)\Phi_{\mu}^{(n,n+\ell-1)}(\zeta_2).$$

Now using (B.1) when $k = \ell - 1$, this becomes

$$\Phi_{\Lambda_{1-i}}^{(n+\ell-1,n+\ell)}(\zeta_2)\Phi_{\Lambda_i}^{(n+\ell-1,n+\ell)}(\zeta_1)\Phi_{\sigma(\mu)}^{(n,n+\ell-1)}(\zeta_2)\Phi_{\mu}^{(n,n+\ell-1)}(\zeta_1)R^{(n,n)}(\zeta).$$

Using the commutativity of $\Phi_{\Lambda_i}^{(n+\ell-1,n+\ell)}(\zeta_1)$ and $\Phi_{\sigma(\mu)}^{(n,n+\ell-1)}(\zeta_2)$ we thus arrive at the equality

$$\begin{aligned} R^{(n+\ell,n+\ell)}(\zeta)&(\Phi_{\Lambda_{1-i}}^{(n+\ell-1,n+\ell)}(\zeta_1)\Phi_{\sigma(\mu)}^{(n,n+\ell-1)}(\zeta_1))(\Phi_{\Lambda_i}^{(n+\ell-1,n+\ell)}(\zeta_2)\Phi_{\mu}^{(n,n+\ell-1)}(\zeta_2)) \\ &= (\Phi_{\Lambda_{1-i}}^{(n+\ell-1,n+\ell)}(\zeta_2)\Phi_{\sigma(\mu)}^{(n,n+\ell-1)}(\zeta_2))(\Phi_{\Lambda_i}^{(n+\ell-1,n+\ell)}(\zeta_1)\Phi_{\mu}^{(n,n+\ell-1)}(\zeta_1))R^{(n,n)}(\zeta). \end{aligned} \quad \text{(B.3)}$$

It is shown in [7] that $\Phi_{\Lambda_i}^{(n+\ell-1,n+\ell)}(\zeta)\Phi_{\mu}^{(n,n+\ell-1)}(\zeta) = (\Phi_{\lambda}^{(n+\ell-1,n+\ell)}(\zeta)\otimes\mathrm{id})$ when restricted to $V(\lambda) \otimes \Omega_{\mu,\Lambda_i;\lambda}$ with $\lambda = \mu + \Lambda_i$. Hence restricting (B.3) to $V(\lambda) \otimes \Omega_{\mu,\Lambda_i;\lambda}$ gives (B.1) with $k = \ell$. This completes the proof.

Appendix C. The perturbative action of intertwiners

In this appendix, we list the perturbative action of the intertwiners used in section 4. We have

$$\begin{aligned} \Phi_{\Lambda_0}^{\Lambda_1 V^{(1)}}(\zeta)v_{\Lambda_0} &= v_{\Lambda_1} \otimes u_1^{(1)} - qf_1 v_{\Lambda_1} \otimes u_0^{(1)}\zeta + \frac{q^3}{[2]}f_0 f_1 v_{\Lambda_1} \otimes u_1^{(1)}\zeta^2 \\ &\quad -\frac{q^4}{[2]}f_1 f_0 f_1 v_{\Lambda_1} \otimes u_0^{(1)}\zeta^3 + \cdots \end{aligned} \quad \text{(C.1)}$$

$$\begin{aligned} \Phi_{2\Lambda_0}^{\Lambda_0+\Lambda_1 V^{(1)}}(\zeta)v_{2\Lambda_0} &= v_{\Lambda_0+\Lambda_1} \otimes u_1^{(1)} - qf_1 v_{\Lambda_0+\Lambda_1} \otimes u_0^{(1)}\zeta \\ &\quad +\frac{q^4}{1-[3]^2}(f_1 f_0 - [3]f_0 f_1)v_{\Lambda_0+\Lambda_1} \otimes u_1^{(1)}\zeta^2 + \cdots \end{aligned} \quad \text{(C.2)}$$

$$\begin{aligned} \Phi_{3\Lambda_0}^{2\Lambda_0+\Lambda_1 V^{(1)}}(\zeta)v_{3\Lambda_0} &= v_{2\Lambda_0+\Lambda_1} \otimes u_1^{(1)} - qf_1 v_{\Lambda_{2\Lambda_0}+\Lambda_1} \otimes u_0^{(1)}\zeta \\ &\quad +\frac{q^5}{[4]+[6]}([3]f_0 f_1 - f_1 f_0)v_{\Lambda_{2\Lambda_0}+\Lambda_1} \otimes u_1^{(1)}\zeta^2 + \cdots \end{aligned} \quad \text{(C.3)}$$

$$\Phi_{2\Lambda_0+\Lambda_1}^{3\Lambda_0 V^{(1)}}(\zeta)v_{2\Lambda_0+\Lambda_1} = v_{3\Lambda_0} \otimes u_0^{(1)} - \frac{q^3}{[3]}f_0 v_{3\Lambda_0} \otimes u_1^{(1)}\zeta$$

$$+\frac{q^5}{[2][3]} f_1 f_0 v_{3\Lambda_0} \otimes u_0^{(1)} \zeta^2 + \cdots \tag{C.4}$$

$$\Phi_{2\Lambda_0+\Lambda_1}^{\Lambda_0+2\Lambda_1 V^{(1)}}(\zeta) v_{2\Lambda_0+\Lambda_1} = v_{\Lambda_0+2\Lambda_1} \otimes u_1^{(1)} - \frac{q^2}{[2]} f_1 v_{\Lambda_0+2\Lambda_1} \otimes u_0^{(1)} \zeta$$

$$+\frac{q^5}{[2]([3][4]-[2])}([4] f_0 f_1 - [2] f_1 f_0) v_{\Lambda_0+2\Lambda_1} \otimes u_1^{(1)} \zeta^2 + \cdots \tag{C.5}$$

$$\Phi_{3\Lambda_0}^{2\Lambda_1+\Lambda_0 V^{(2)}}(\zeta) v_{3\Lambda_0} = v_{2\Lambda_1+\Lambda_0} \otimes u_2^{(2)} - \frac{q^{3/2}}{[2]^{1/2}} f_1 v_{2\Lambda_1+\Lambda_0} \otimes u_1^{(2)} \zeta + \frac{q^2}{[2]} f_1^2 v_{2\Lambda_1+\Lambda_0} \otimes u_0^{(2)} \zeta^2$$

$$+\frac{q^5}{[4][3]-[2]}([4] f_0 f_1 - [2] f_1 f_0) v_{2\Lambda_1+\Lambda_0} \otimes u_2^{(2)} \zeta^2 + \cdots \tag{C.6}$$

$$\Phi_{\Lambda_0}^{(1,2)}(\zeta)(u_1^{(1)} \otimes v_{\Lambda_0}) = v_{\Lambda_1} \otimes u_2^{(2)} - \frac{q^{3/2}}{[2]^{1/2}} f_1 v_{\Lambda_1} \otimes u_1^{(2)} \zeta + \frac{q^4}{[2]} f_0 f_1 v_{\Lambda_1} \otimes u_2^{(2)} \zeta^2 \cdots \tag{C.7}$$

$$\Phi_{\Lambda_0}^{(1,2)}(\zeta)(u_0^{(1)} \otimes v_{\Lambda_0}) = \frac{q^{-1/2}}{[2]^{1/2}} v_{\Lambda_1} \otimes u_1^{(2)} - q f_1 v_{\Lambda_1} \otimes u_0^{(2)} \zeta$$

$$+\frac{q^{7/2}}{[2]^{3/2}} f_0 f_1 v_{\Lambda_1} \otimes u_1^{(2)} \zeta^2 + \cdots . \tag{C.8}$$

All other intertwiners we need are given by a $(f_i, \Lambda_j, u_\ell^{(n)}) \leftrightarrow (f_{1-i}, \Lambda_{1-j}, u_{n-\ell}^{(n)})$ symmetry, for example the expansion

$$\Phi_{\Lambda_1}^{\Lambda_0 V^{(1)}}(\zeta) v_{\Lambda_1} = v_{\Lambda_0} \otimes u_0^{(1)} - q f_0 v_{\Lambda_0} \otimes u_1^{(1)} \zeta + q^3 \frac{1}{[2]} f_1 f_0 v_{\Lambda_0} \otimes u_0^{(1)} \zeta^2$$

$$-q^4 \frac{1}{[2]} f_0 f_1 f_0 v_{\Lambda_0} \otimes u_1^{(1)} \zeta^3 + \cdots$$

follows from (C.1) under this symmetry. This symmetry is one of the benefits of using a principal evaluation module.

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